New realizations of the dual space of SU(3)

1993 J. Phys. A: Math. Gen. 26 1625

(http://iopscience.iop.org/0305-4470/26/7/019)

J. Phys. A: Math. Gen. 26 (1993) 1625–1633. Printed in the UK

New realizations of the dual space of $SU(3)$

J S Prakash and H S Sharatchandra†

The Institute of Mathematical Sciences, Madras-600 113, India

Received 10 August 1992, in final form 1 December 1992

Abstract. The logic of constructing and interrelating classical realizations of the dual space of $SO(3)$ is exhibited. Analogies with different methods of quantization of gauge theories are pointed out. The analysis is applied to $SU(3)$ to obtain existing realizations and to construct new ones. Gelfand–Zetlyn basis for the irreducible representations of $SU(3)$ is explicitly realized using polynomials in four variables and positive or negative integral powers of a fifth variable. Another realization uses a spinor of $SO(6)\times SO(3,1)$. These are the analogues of Schwinger–Bargmann construction for $SU(2)$.

1. Introduction

In many problems involving compact Lie groups, it is found that only the $SU(2)$ case can be explicitly handled. This is despite the exhaustive study of representation theory. The stumbling block is the absence [1, 2] (except for $SU(2)$) of explicit and convenient realizations of the dual of Lie groups. The dual of a group is defined as the set of all equivalence classes of irreducible unitary representations of the group. We refer to any concrete realization of the dual as a *model* of the group in question (slightly altering the terminology of Gelfand *et al* [2]). Also we call the space on which the model is realized, the 'model space'.

A glaring example of the difference between $SU(2)$ and other compact groups is that of Clebsch–Gordan (CG) coefficients, which are needed in many applications. In $SU(2)$ case we have a simple and elegant generating function for these coefficients [3] and this is made possible because of a convenient model.

There are elegant models for other compact Lie groups using different approaches. See, for instance [4–9]. But these do not appear to be tangible enough for specific applications, e.g. the CG coefficients. To clarify this, we may compare the situation with the $SO(3)$ case, which has three well known models. The model spaces for these three are: type I—homogeneous harmonic polynomials in three real variables; type II—spherical harmonics, i.e. functions on a 2-sphere; and type III—homogeneous, analytic functions in two variables which is also related to the spinor representation of $SO(3)$ and to the boson calculus. In contrast, in the $SU(3)$ case, only the analogues of type I and type II are presented in the literature. Note that it is the type III model space which gives the generating function [3, 10] for the CG coefficients of $SO(3)$ (and $SU(2)$ of course). Therefore an analogous construction is important for other groups.

In this paper we carry out this construction for $SU(3)$. We obtain a model which uses non-negative integral powers of four variables and (negative or positive) integral

† E-mail address: sharat@imsc.ernet. in

powers of a fifth variable. This model is connected to the boson calculus for $SU(3)$ which we have developed from different considerations earlier [11]. We also obtain another model which uses spinor representation of the group $SO(6)\times SO(3,1)$. In fact we obtain this model exactly following Cartan's [12] development of the theory of spinors for $SO(n)$ group, n arbitrary, starting from the $n=3$ case. We expect our techniques can be generalized [4, 9] to other groups.

As mentioned above, both the earlier models and models we construct for $SU(3)$ have exact parallels to the well known models of $SO(3)$. To emphasize and exploit this analogy, we discuss these classical models of $SO(3)$ from a unified view point in section 2. We exhibit the logic of constructing and interrelating models and the close analogy with the techniques employed in gauge theories. We apply this analysis to $SU(3)$ in section 3. We recover the models available in the literature. We then pursue the analysis in sections 4 and 5 to construct type III models.

2. Classical models of $SO(3)$ from a unified view point

$SO(3)$ $(=SO(3,R))$ is defined as the group of real 3×3 matrices of determinant $+1$ acting on 3-tuple of real (or complex) numbers $x=(x_1, x_2, x_3)$ and leaving the quadratic form

$$x^2=x_1^2+x_2^2+x_3^2 \tag{2.1}$$

invariant. This gives the defining representation, where (x_1, x_2, x_3) are regarded as components of a vector of a three-dimensional vector space V in some orthonormal basis. All other irreducible representations (IR) may be realized using direct products of V. In fact the IR labelled by the angular momentum quantum number l may be realized on the space of homogeneous polynomials in x_i, $i=1,2,3$ of degree l. Nevertheless, the space P of all homogeneous polynomials in $\{x_i\}$ does not give a model space of $SO(3)$, because various IRs appear with different multiplicities. The underlying reason is the invariant combination (2.1). A way to overcome this problem is as follows. The operator

$$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2} \tag{2.2}$$

commutes with the angular momentum operators J_i, $i=1,2,3$ and acting on homogeneous polynomials gives again homogeneous polynomials. Therefore, we may realize the IR on the subspace H of homogeneous polynomials annihilated by ∇^2, i.e. on the harmonic polynomials. Indeed, H provides a model space of $SO(3)$. Homogeneous polynomials of the type $x^2 f(x_1, x_2, x_3)$ are absent as a consequence of the harmonicity condition. We call this the type I model.

This approach has been generalized to other compact Lie groups by Moshinsky [4]. Though it is easy and elegant to characterize type I models, it is not the best suited for applications. As a consequence of the constraints like (2.2) on the function space, it is difficult to realize the basis explicitly.

An alternate way of avoiding multiplicity of the IRs is to impose the constraint

$$x^2\equiv x_1^2+x_2^2+x_3^2=K \tag{2.3}$$

where K is a constant, on the invariant (2.1). Choosing $K=1$ for instance, we get a model realized on functions on a 2-sphere (i.e. spherical harmonics). This is type II model.

Kramers [7] for $SU(3)$, and Gelfand *et al* [9] for all compact Lie groups, have constructed type II models using certain symmetric spaces. In type II models, the function space is unrestricted, but one has to contend with constraints such as (2.3) on the variables. This complicates the basic functions.

Remarkably, there is a way of removing disadvantages of both type I and II models, which, moreover, has a deeper mathematical significance. For this, one has to first choose the singular case $K=0$ for the constraint (2.3). Of course, we are now forced to choose x_i to be complex variables. (The action of $SO(3)$ is still with real coefficients on these complex variables.) However we now choose functions analytic in each x_i. Thereby, each IR is again realized only once.

With $K=0$, we have a non-compact space, the quadric cone, instead of the compact 2-sphere ($K>0$). We can now solve the constraint (2.3) preserving homogeneity. To see this, we rewrite (2.3) as,

$$(x_1-ix_2)(x_1+ix_2)+x_3^2=0$$

which is identically satisfied by setting,

$$x_1+ix_2=\zeta^2 \qquad x_1-ix_2=\eta^2 \qquad x_3=i\zeta\eta. \tag{2.4}$$

As ζ and η take all complex values, every point of the cone is covered, twice, with (ζ,η) and $(-\zeta,-\eta)$ representing the same point. The model space is given by arbitrary functions in ζ and η, invariant under $(\zeta,\eta)\to(-\zeta,-\eta)$.

It is well known that (ζ,η) transforms like the spinorial representation of $SO(3)$. When we consider a linear $SU(2)$ transformation of (ζ,η), x_i, $i=1,2,3$ defined via (2.4) transform linearly under the related $SO(3)$ matrix. There is an intimate relation between the isotropy condition $K=0$ and the spinorial representation as analysed in detail by Cartan [12].

The IRs of $SO(3)$ are realized on homogeneous polynomials of even degree in ζ and η. (By considering also the odd degree, we get a model of $SU(2)$.) ζ and η are eigenstates of J_z with eigenvalues $\pm\frac{1}{2}$ respectively. Hence the basis states $|jm\rangle$ of the IRs are simply represented by the monomials $\zeta^{j+m}\eta^{j-m}$ up to a normalization factor. This gives a type III model.

The basis vectors can be explicitly constructed and are very simple in type III models. Therefore this is well suited for applications.

The analysis given above for obtaining and interrelating the models is exactly parallel to certain techniques used in the quantization of gauge theories. Consider quantum electrodynamics (without matter), as an example. The state of the system is described by the wavefunctional $\psi\,[A]$ of the vector potential $A_i(X)$, $i=1,2,3$. However, a single physical state may be described by many different wavefunctionals. In fact, two wavefunctionals which agree on the transverse part of every vector potential describe the same physical state. In order to remove this multiplicity, we may choose a subspace of the space of functions. This subspace is given by those wavefunctionals which satisfy the Gauss constraint:

$$\sum_i \frac{\partial}{\partial X_i}\frac{\delta}{\delta A_i(X)}\psi[A]=0 \tag{2.5}$$

for every X. Here $\delta/\delta A_i$ is the functional derivative. (2.5) is the analogue of (2.2). This corresponds to type I model.

It is difficult to work with constrained functionals as in the earlier case. Therefore a 'gauge fixing' procedure is often adopted. Because wavefunctionals which agree on the transverse part of every vector potential are equivalent, we choose a representative from among the vector potentials with the same transverse part. An example is the 'Coulomb gauge' which uses only vector potentials $A(\bar{X})$ which satisfy

$$\sum_i \frac{\partial}{\partial X_i} A_i(X) = 0. \tag{2.6}$$

This is the analogue of (2.3). Now all wavefunctionals of $A(X)$ (satisfying (2.6)) are permitted. This corresponds to type II model.

Instead of working with constrained vector potentials such as (2.6), one may try to solve the constraint and use arbitrary wavefunctionals of the independent degrees of freedom. For example, one may use arbitrary functionals of transverse part $A^T(X)$ of the vector potential. This would correspond to type III models.

3. Models of $SU(3)$

We now use the analysis of section 2 to construct models of $SU(3)$. We recover the models that have been presented in the literature. They correspond to type I and type II models.

In the case of $SO(3)$, all IRs could be built using one triplet $\{x_i\}$ alone. This is not possible in the case of $SU(3)$. We need [4] at least two triplet representations of $SU(3)$. An easy way to see this is as follows. A general Young tableaux for $SU(3)$ has two rows of boxes, at the most. Consider Young tableau with just one row. The corresponding tensors are totally symmetric in the indices (each of which can take values 1, 2 or 3). Such a tensor is simply represented by homogeneous polynomials in (z^1, z^2, z^3) transforming like $\underset{\sim}{3}$ of $SU(3)$. Such a polynomial is automatically totally symmetric. In the case of tableau with two rows, the corresponding tensors are antisymmetric in indices along a column. To be able to construct polynomials with this antisymmetry, we need one more triplet.

Instead of using a pair of $\underset{\sim}{3}$ representations, one may use [5, 6, 8] one $\underset{\sim}{3}$ and one $\underset{\sim}{3}^*$. Though equivalent, the latter turns out to be more convenient, and will be used below.

The IRs of $SU(3)$ are uniquely realized [13] on tensors $T^{i_1 i_2 \ldots i_M}_{j_1 j_2 \ldots j_N}$ $(i, j = 1, 2, 3)$ which are totally symmetric in the contravariant indices and in the covariant indices and are moreover traceless with respect to the contraction of any contravariant with any covariant index. Such tensors are easily mimicked [5, 6, 14] by polynomials homogeneous in (z^1, z^2, z^3) and in (w_1, w_2, w_3) of degrees M and N respectively.

$$\begin{aligned} P^M_N(z^i, w_j) \equiv\; & z^{i_1} z^{i_2} \ldots z^{i_M} w_{j_1} w_{j_2} \ldots w_{j_N} \\ & + \alpha_1 z \cdot w(\delta^{i_1}_{j_1} z^{i_2} \ldots z^{i_M} w_{j_2} \ldots w_{j_N} + \text{permutations}) \\ & + \alpha_2 (z \cdot w)^2 (\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} z^{i_3} \ldots z^{i_M} w_{j_3} \ldots w_{j_N} + \text{permutations}) \\ & + (\text{higher powers of } z \cdot w). \end{aligned} \tag{3.1}$$

z^i and w_i transform like $\underset{\sim}{3}$ and $\underset{\sim}{3}^*$ respectively. The coefficients are to be chosen so as to satisfy the tracelessness condition. This algebraic constraint may be equivalently stated as a differential constraint, [4, 5]

$$\sum_{i=1} \frac{\partial}{\partial z^i} \frac{\partial}{\partial w_i} P^m_n = 0. \tag{3.2}$$

This is a type I model. We may also identify $w_i^* = z^i$ in this construction.

It is cumbersome to compute the coefficients $\alpha_1, \alpha_2, \ldots$ and to work [5, 6, 14] with the polynomials (3.1). A way out is to note that the functions (3.1) are uniquely distinguished by their values on the constrained surface,

$$\sum_{i=1} z^i w_i = K \tag{3.3}$$

where K is any complex number. The tensorial property of (3.1) is not affected because the constraint (3.3) is invariant under $SU(3)$ transformations. For $K \neq 0$, the constrained space is a quadric in six-dimensional affine space. With $w_i^* = z^i$, we have a 5-sphere. Considering all functions, complex analytic and real respectively, on these spaces, we get a type II model [7, 9, 15].

4. A new model: explicit basis using polynomials in five variables

We now proceed to construct type III models for $SU(3)$ which are not available in the literature. As in the $SO(3)$ case, we make the singular choice $K = 0$ in (3.3), which gives the quadric cone [16]. Now all terms, except the first, in the tensors (3.1) drop out. Therefore we may obtain a basis for the model space by considering arbitrary polynomials homogeneous in $\{z^i\}$ and $\{w_j\}$ and identifying those that agree on the surface $z \cdot w = 0$. There is a very easy way of solving this constraint to get a simple basis for the IRs. We simply eliminate w_3 in favour of the other variables:

$$w_3 = \frac{z^1 w_1 + z^2 w_2}{z^3}. \tag{4.1}$$

This preserves homogeneity in (w_1, w_2) and (z^1, z^2, z^3) separately.

In order to obtain an explicit basis for each IR, we proceed as follows. An IR may be labelled by the numbers (M, N). Here M is the number of columns with one box each and N is the number of columns with two boxes in each column. Noting that a 2-column Young tableau corresponds to the $\underset{\sim}{3}^*$ representation, we see that the IR (M, N) can be realized in the space of polynomials P_N^M spanned by the monomials

$$(z^1)^{M_1}(z^2)^{M_2}(z^3)^{M_3}(w_1)^{N_1}(w_2)^{N_2}(w_3)^{N_3} \tag{4.2}$$

with

$$M_1 + M_2 + M_3 = M \qquad N_1 + N_2 + N_3 = N. \tag{4.3}$$

This space also contains some other IRs $(M_1' N_2')$ with $M' < M$, $N' < N$. This is because of the possibility of forming the invariant combination $z \cdot w$. This also means that on the surface $z \cdot w = 0$, these other IRs drop out.

We now regard w_3 as a function of the other five variables, equation (4.1). To get an explicit basis now, we note that (z^1, z^2) and (w_1, w_2) transform like $\underset{\sim}{2}$ and $\underset{\sim}{2}^*$ (which is equivalent to $\underset{\sim}{2}$) representations of the isospin $SU(2)$ subgroup of $SU(3)$. The combination $(z^1 w_1 + z^2 w_2)$ in (4.1) is an isospin singlet as it should be, because z^3 and w_3 are also singlets. This combination suggests a change of basis in (4.2) as follows.

Consider the generating function,

$$(az^1 + bz^2)^{2j}$$

of homogeneous polynomials in z^1 and z^2 of degree $2j$. Irreducible representation corresponding to isospin j is realized on this space. The coefficient of $a^p b^q$ represents (up to a normalization), the state $|jm\rangle$ where

$$2j = p + q \qquad 2m = p - q. \tag{4.4}$$

Thus the generating function $(az^1 + bz^2)^{2j}$ $(cw_1 + dw_2)^{2j'}$ corresponds to the direct product of two IRs j and j' of isospin. We may now make a change to the coupled basis. We get each of the IRs $j'' = (j + j'), (j + j' - 1), \ldots,$ and $|j - j'|$ once. The generating function for a specific IR j'' is

$$(z^1 w_1 + 2^2 w_2)^k (az^1 + bz^2)^{2j-k} (aw_2 - bw_1)^{2j'-k} \tag{4.5}$$

where the non-negative integer k is given by

$$k = j + j' - j'' \qquad \min(2j, 2j') \geqslant k \geqslant 0. \tag{4.6}$$

A basis state $|jj'j''m''\rangle$ of the coupled basis is represented by that polynomial in z^1, z^2, w_1 and w_2 which is the coefficient of the monomial $a^p b^q$ in (4.5). The choice of p and q for given j'' and m'' is as in (4.4).

It is convenient to relabel the powers in (4.5) as follows. Define,

$$R_1 = j' + j'' - j \qquad R_2 = j'' + j - j' \qquad R_3 = j + j' - j''. \tag{4.7}$$

These are the variables which appear in the Regge symmetries of the $3-j$ symbol. For any given j and j' and all allowed values of j'', R_i's are non-negative integers. Further, every choice of j and j' and any j'' of the direct product is uniquely reproduced as R_i, $i = 1, 2, 3$ take all non-negative integer values. Using these variables, (4.5) is

$$(z^1 w_1 + z^2 w_2)^{R_3} (az^1 + bz^2)^{R_2} (aw_2 - bw_1)^{R_1}. \tag{4.8}$$

Now we consider the subspace spanned by

$$M_1 + M_2 = 2j \qquad N_1 + N_2 = 2j' \tag{4.9}$$

in the space P_N^M of the polynomials (4.2). In this subspace we make a change of basis (4.7). We get basis vectors as coefficients of the monomials $a^p b^q$ in

$$(z^1 w_1 + z^2 w_2)^{R_3} (az^1 + bz^2)^{R_2} (aw_2 - bw_1)^{R_1} (z^3)^{M_3} (w_3)^{N_3}. \tag{4.10}$$

This basis is equivalent to (4.2) as R_1, R_2, R_3, M_3 and N_3 range over non-negative integers. From (4.3) and (4.6) we see that an IR (M, N) of $SU(3)$ may be realized in the space of polynomials spanned by all non-negative integers R_1, R_2, R_3, M_3 and N_3 such that

$$R_2 + R_3 + M_3 = M \qquad R_1 + R_3 + N_3 = N. \tag{4.11}$$

We are now in a position to extract explicit and distinct basis vectors on the constrained surface $z \cdot w = 0$. Making the replacement (4.1) in (4.7) we get

$$(z^1 w_1 + z^2 w_2)^{R_3 + N_3} (az^1 + bz^2)^{R_2} (aw_2 - bw_1)^{R_1} (z^3)^{M_3 - N_3} \tag{4.12}$$

where R_1, R_2, R_3, M_3, $N_3 \geqslant 0$.

We see that (4.11) is indistinguishable for distinct (non-negative) values of M_3, N_3 and R_3 such that

$$K = R_3 + N_3 \qquad H = R_3 + M_3 \tag{4.13}$$

are the same. This is the way that equivalent IRs contained in the space P_N^M get identified on the surface $z \cdot w = 0$. Using the new variables (4.12), a given IR (M, N) is spanned by

$$\left(\frac{z^1 w_1 + z^2 w_2}{z^3}\right)^K (az^1 + bz^2)^{R_2} (aw_2 - bw_1)^{R_1} (z^3)^H \tag{4.14}$$

where R_1, R_2, H and K are non-negative integers subject to the constraints

$$R_2+H=M \qquad R_1+K=N. \tag{4.15}$$

Considering all non-negative values of R_1, R_2, K and H subject to the constraints (4.15) and collecting the coefficients of various monomials $a^p b^q$ in (4.14), we get an explicit basis for the IRs of $SU(3)$. The states are labelled by p, q, R_1, R_2, H, K subject to the constraint (4.16). In fact we have realized the basis used in the quark model (i.e. the Gelfand–Zetlyn scheme). As we have used the coupled basis for isospin, we have

$$2I=R_1+R_2=p+q \qquad 2I_3=p-q \tag{4.16}$$

where I is the isospin quantum number and I_3 is the eigenvalue of the third component of the isospin. Further noting that z^3 has strangeness $S=-1$, whereas (z^1,z^2), (w_1,w_2) have $S=0$, we get

$$S=K-H. \tag{4.17}$$

5. Model space using spinor of $SO(6)\times SO(3,1)$

In case of $SO(3)$, type III model involves a doublet (z^1,z^2) of complex variables transforming as the spinor of $SO(3)$. We now give an analogous construction for $SU(3)$.

In section 4, we solved the constraint $z\cdot w=0$ by regarding w_3 as a dependent variable. But this has the disadvantage that the variables z_1^1, z_2^2, z^3, w_1 and w_2 used in the model have nonlinear transformations under $SU(3)$. Taking cue from the $SO(3)$ case, we want to now solve this constraint in a different way. In fact we have to simply use the analysis of Cartan [12] to generalize the notion of spinors of $SO(3)$ to an arbitrary orthogonal group.

We may regard $z\cdot w=0$ as the condition that the vector $(z^1,z^2,z^3,w_1,w_2,w_3)$ in the Euclidean space $E(6)$ be isotropic [12] (equivalently 'null'). (Using variables ξ_i, where $z_i=\xi_i+i\xi_{i+3}$, $w_i=\xi_i-i\xi_{i+3}$, $i=1,2,3$, we get the standard form $\Sigma_{i=1}^6\,\xi_i^2=0$). Such vectors can be constructed using bilinears of spinors. In our case of six-dimensional vectors, the results are as follows (see [12] p 117). The semi- (equivalently 'chiral') spinors have dimension $2^{6/2-1}=4$. Consider two such, ξ_α and η_α, $\alpha=0,1,2,3$. Then the following vector, formed as bilinear in ξ and η, is isotropic.

$$\begin{aligned}
z^1&=\begin{vmatrix}\xi_0 & \eta_0\\ \xi_1 & \eta_1\end{vmatrix} \qquad z^2=\begin{vmatrix}\xi_0 & \eta_0\\ \xi_2 & \eta_2\end{vmatrix} \qquad z^3=\begin{vmatrix}\xi_0 & \eta_0\\ \xi_3 & \eta_3\end{vmatrix}\\
w_1&=\begin{vmatrix}\xi_2 & \eta_2\\ \xi_3 & \eta_3\end{vmatrix} \qquad w_2=\begin{vmatrix}\xi_3 & \eta_3\\ \xi_1 & \eta_1\end{vmatrix} \qquad w_3=\begin{vmatrix}\xi_1 & \eta_1\\ \xi_2 & \eta_2\end{vmatrix}
\end{aligned} \tag{5.1}$$

where the bars stand for determinant of the 2×2 array. ξ_α (and similarly η_α) may be regarded as the homogenous coordinates of a point ξ in a projective space of three dimensions. The components (z^i,w_i) then have the interpretation as the Plücker coordinates of the line which joins the two points ξ and η.

It is useful to rewrite (5.1) using the terminology of three-dimensional vectors:

$$\boldsymbol{z}=\xi_0\boldsymbol{\eta}-\eta_0\boldsymbol{\xi} \qquad \boldsymbol{w}=\boldsymbol{\xi}\times\boldsymbol{\eta} \tag{5.2}$$

where $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$ etc. Thus $\boldsymbol{z}$ is in the plane spanned by $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ whereas $\boldsymbol{w}$ is perpendicular to this plane. Now it is also easy to verify that every isotropic 6-vector is obtained as we consider all complex valued ξ_α and η_α. We may simply choose the special case $\eta_0=0$ and $\xi_0=1$. Then, $\boldsymbol{z}=\boldsymbol{\eta}$, which may be chosen arbitrarily. Further for any $z \neq 0$, every vector perpendicular to it can be obtained by considering $\boldsymbol{\xi} \times \boldsymbol{\eta}$, with $\boldsymbol{\xi}$ an arbitrary vector. (For the special case $z = 0$, we may choose $\xi_0 = \eta_0 = 0$ and $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as arbitrary vectors.)

It is to be observed that we have solved for six variables, (z^i, w_i) involving one constraint using eight variables $(\xi_\alpha, \eta_\alpha)$. Therefore it is to be expected that many distinct pairs of spinors ξ_α and η_α reproduce a single isotropic vector. It is also easy to characterize such equivalent pairs. $\boldsymbol{w}$ is perpendicular to the plane spanned by $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Therefore we may hope to reproduce the same $\boldsymbol{w}$ by choosing other vectors $\boldsymbol{\xi}'$ and $\boldsymbol{\eta}'$ in the same plane, i.e.

$$\boldsymbol{\xi}' = \alpha\boldsymbol{\xi} + \beta\boldsymbol{\eta} \qquad \boldsymbol{\eta}' = \gamma\boldsymbol{\xi} + \delta\boldsymbol{\eta} \tag{5.3}$$

with complex numbers α, β, γ and δ. We have $\boldsymbol{\xi}' \times \boldsymbol{\eta}' = \boldsymbol{\xi} \times \boldsymbol{\eta}$ if

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1. \tag{5.4}$$

Further if (ξ_0, η_0) also transforms like (ξ_i, η_i), i.e.

$$\begin{pmatrix} \xi_0' \\ \eta_0' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} \tag{5.5}$$

then $z = \xi_0 \boldsymbol{\eta} - \eta_0 \boldsymbol{\xi}$ is also left invariant. Thus if we regard the two $SO(6, \mathbb{C})$ spinors ξ_α and η_α as four doublets, $u_\alpha = (\xi_\alpha, \eta_\alpha)$, $\alpha = 1, 2, 3$ or 0, then a common $SL(2, \mathbb{C})$ transformation of these four doublets leaves (z^i, w_i) invariant.

Each doublet u_α, $\alpha = 1, 2, 3$ or 4 may be therefore regarded as a spinor of the Lorentz group $SO(3, 1)$. Thus the eight objects $(\xi_\alpha, \eta_\alpha)$ form a spinor of the direct product $SO(6) \times SO(3, 1)$. We have expressed an arbitrary isotropic 6-vector as a bilinear of this spinor. The vector so obtained is invariant under $SO(3, 1)$ transformations of the spinors.

The $SU(3)$ group under which z and w transform like $\underset{\sim}{3}$ and $\underset{\sim}{3}^*$ is a subgroup of $SO(6)$ under which $z \cdot w = 0$ is invariant. Therefore the spinors ξ_α and η_α also transform as representations of $SU(3)$. It is easy to read off these transformations from (5.3). If $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ each transform as $\underset{\sim}{3}$ and ξ_0 and η_0 as singlets of $SU(3)$, then z and w have the right transformation properties.

Thus our model using spinors is closely related to Moshinsky's (type I) model [4] which uses a pair of $\underset{\sim}{3}$ representations. We have in addition, two singlets ξ_0 and η_0. As a result, we have a larger symmetry $SO(6) \times SO(3, 1)$. This addition is not just a redundancy. The advantage is that we can now construct explicit basis vectors using objects which transform linearly under $SU(3)$. One way of doing this is to simply substitute for z^1, z^2, z^3, w_1 and w_2 in (4.14), bilinear expressions in spinors. This basis will be developed further elsewhere.

6. Discussion

Our aim in this paper is to generalize certain techniques available for $SU(2)$ to $SU(3)$ (and other Lie groups) so as to make them as accessible as $SU(2)$ is for applications. For this purpose we construct the classical models of $SO(3)$ from first principles and show how they are related to one another. We point out close analogy with different methods of quantization of gauge theories. We apply this analysis to $SU(3)$. We recover models available in the literature and construct new models. Our new models are the analogues of the Schwinger–Bargmann model of $SO(3)$ which uses the spinorial representation. We have simple and explicit realizations of basis vectors of irreducible representations of $SU(3)$, analogous to the monomial basis of Bargmann.

The analogy with gauge theories can be put to use in other ways. For instance, the BRST techniques may be employed and also a cohomology interpretation may be given.

We will use the techniques of this paper to construct a generating function for the Clebsch–Gordan coefficients of $SU(3)$ elsewhere.

Acknowledgments

One of us (HSS) thanks Subhashis Nag for some clarifications.

References

[1] Biedenharn L C 1963 *J. Math. Phys.* **4** 436

[2] Bernstein I N, Gelfand I M and Gelfand S I 1976 *Proc. Petrovskij Sem.* **2** 3
Reprinted in Gelfand I 1988 *Collected Works Vol. II* (Berlin: Springer) p 464

[3] Schwinger J 1952 On angular momentum (US Atomic Energy Commission) NYO-3071 unpublished
Bargmann V 1962 *Rev. Mod. Phys.* **34** 829
Both articles are reprinted in *Quantum Theory of Angular Momentum* ed L C Biedenharn and H Van Dam (New York: Academic)

[4] Moshinsky M 1962 *Rev. Mod. Phys.* **34** 813; 1963 *J. Math. Phys.* **4** 1128

[5] Chew C K and Sharp R T 1986 *Can. J. Phys.* **44** 2789

[6] Jasselette P 1967 *Nucl. Phys.* **1** 521, 529; 1980 *J. Phys. A: Math. Gen.* **13** 226

[7] Krämer M 1979 *Arch. Math.* **33** 76

[8] Biedenharn L C and Flath D 1984 *Commun. Math. Phys.* **93** 143
Bracken A J and MacGibbon J H 1984 *J. Phys. A: Math. Gen.* **17** 2581
Bracken A J 1984 *Commun. Math. Phys.* **94** 371
Deenen J and Quesne C 1986 *J. Phys. A: Math. Gen.* **19** 3463
Van Der Jeugt J 1985 *Lett. Nuovo Cimento* **43** 229

[9] Gelfand I M and Zelevinskij A V 1984 *Funkts. Anal. Prilozh.* **18** (3) 14 reprinted in *Collected Works* ibid.

[10] Beg M A B and Ruegge H 1985 *J. Math. Phys.* **6** 677
Kleima D 1985 *Nucl. Phys.* **70** 577–602

[11] Smorodinskii Ya A and Shelepin L A 1972 *Usp. Fiz. Nauk* **106**

[12] Prakash J S and Sharatchandra H S 1992 Optimal boson calculus for $SU(3)$: unique resolution of the weight lattice by random integers *Preprint* No. imsc-92/05

[13] Cartan E 1966 *The Theory of Spinors* (Paris: Hermann)

[14] Coleman S 1964 *J. Math. Phys.* **5** 1343

[15] Mukunda N and Pandit L K 1965 *J. Math. Phys.* **6** 746; 1965 *Prog. Theor. Phys.* **34** 46

[16] Zelobenko D P 1973 *Compact Lie Groups and their Representations* (Providence, RI: American Mathematical Society) p 123